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E. W. Cheney

A. A. Goldstein

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MEAN-SQUARE APPROXIMATION BY GENERALIZED

RATIONAL FUNCTIONS

by

E. W. Cheney\*  
University of Texas

and

A. A. Goldstein\*\*  
University of Washington

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# ABSTRACT

The problem of numerical analysis to which this study is directed is that of determining an optimum approximation (in the least squares sense) to a given function  $f$  by a function of the form  $p/q$ , where  $p$  and  $q$  are confined to certain prescribed linear spaces. The analogous approximation problem employing the *uniform* norm has received much recent attention. See, for example, [1, 2, 3, 4, 5, 9]. To our knowledge no investigation of the present problem has appeared.

The problem of numerical analysis to which this study is directed is that of determining an optimum approximation (in the least squares sense) to a given function  $f$  by a function of the form  $p/q$ , where  $p$  and  $q$  are confined to certain prescribed linear spaces. The analogous approximation problem employing the *uniform* norm has received much recent attention. See, for example, [1, 2, 3, 4, 5, 9]. To our knowledge no investigation of the present problem has appeared.

The exact setting of the problem will be as follows. We consider the linear space,  $C[a,b]$ , of all continuous real-valued functions defined on a fixed, closed, interval  $[a,b]$ . In many of the results, the reader will observe that the domain  $[a,b]$  can be replaced by an arbitrary compact measure space. In  $C[a,b]$  we consider two norms

$$\|f\|_{\infty} = \max_{a \leq x \leq b} |f(x)|$$

$$\|f\|_2 = \left\{ \int_a^b [f(x)]^2 w(x) dx \right\}^{1/2}.$$

In the second equation,  $w$  denotes a continuous, positive, function which is defined on  $[a,b]$  and remains unchanged in the discussion. Corresponding to the second norm there is an inner product defined in  $C[a,b]$  by the equation

$$\langle f, g \rangle = \int_a^b f(x) g(x) w(x) dx.$$

In the space  $C[a,b]$  two finite dimensional linear subspaces,  $P$  and  $Q$ , are prescribed. Of the subspace  $Q$ , we demand that it contain at least one element which remains positive throughout  $[a,b]$ . It is convenient to write  $q > 0$  for the more exact assertion

$\inf_{a \leq x \leq b} q(x) > 0$ , and to put  $Q^+ = \{q: q \in Q, q > 0\}$ . Next we form a class  $R$  in  $C[a,b]$  by writing

$$R = R(P,Q) = \{p/q: p \in P, q \in Q^+\}$$

and we contemplate the approximation of arbitrary elements in  $C[a,b]$  by the elements of  $R$ . The functions in  $R$  are called *generalized rational functions*. A great store of knowledge exists about the important special case when  $P$  consists of all polynomials of degree  $\leq n$  and  $Q$  consists of all polynomials of degree  $\leq m$ . The elements of  $R$ , being then of the form  $\sum_{i=0}^n a_i x^i / \sum_{i=0}^m b_i x^i$ , are termed *ordinary rational functions*. The treatise [6] is the principal source of information about approximation by ordinary rational functions. When our arguments must be specialized to this case, we shall replace  $R$  by the symbol  $R_m^n[a,b]$ . Thus:

$$R_m^n[a,b] = \{p/q : p(x) = \sum_{i=0}^n a_i x^i, q(x) = \sum_{i=0}^m b_i x^i, q > 0\}.$$

We shall say that a given element  $r_0$  of  $R$  is a *best  $L^2$  approximation* of a function  $f \in C[a,b]$  if the inequality

$$\|f - r_0\|_2 \leq \|f - r\|_2$$

is satisfied by each element  $r \in R$ .

The situation regarding the *existence* of a best approximation  $r_0$  is very similar to that involving the uniform norm. Namely, for *generalized* rational functions, a particular function  $f$  may fail to possess a best approximation, while for *ordinary* rational functions the existence can be proved. It is possible to prove existence also in the case of *trigonometric* rational functions:

$$r(x) = \frac{\sum_{k=0}^n (a_k \cos kx + b_k \sin kx)}{\sum_{k=0}^m (c_k \cos kx + d_k \sin kx)} .$$

The existence theorem for best approximations by ordinary rational functions is due to Walsh, both in the uniform case and in the case of various integral means. These results are conveniently found in [6]. The first theorem below is an existence theorem in a general setting which includes the ordinary rational functions, the trigonometric rational functions, and presumably still other cases. The theorem is also applicable to a wide variety of norms. Some preliminary definitions are necessary.

Let  $N$  be a monotone norm defined on  $C[a,b]$ . By *monotone* we mean that

$$|f(x)| \leq |g(x)| \text{ (all } x) \Rightarrow N(f) \leq N(g).$$

We assume that  $N$  may be extended (without losing monotonicity) to functions of the form  $f_S$

$$f_S(x) = \begin{cases} f(x) & x \in S \\ 0 & x \notin S \end{cases}$$

where  $f \in C[a,b]$  and  $S$  is any closed subset of  $[a,b]$ .

It is convenient to define  $N_S$  by  $N_S(f) = N(f_S)$ . For example, if  $N(f) = \max_{a \leq x \leq b} |f(x)|$  then  $N_S(f) = \max_{x \in S} |f(x)|$ . Or, if  $N(f) = \int_a^b |f|$  then  $N_S(f) = \int_S |f|$ . A norm which possesses such an extension is said to be of type (A). The support of a function  $q$  is the set  $\{x: q(x) \neq 0\}$ . The constant function with value  $\beta$  on  $[a,b]$  will be denoted by  $\beta$ .

The subspaces  $P, Q$  and the norm  $N$  are said jointly to have property (C) if

$$\left. \begin{array}{l} p \in P \\ q \in Q \\ f \in C[a,b] \\ N_S(f-p/q) \leq \lambda \text{ for all closed} \\ \text{sets } S \text{ contained in the sup-} \\ \text{port of } q. \end{array} \right\} \begin{array}{l} \text{there exist } p_0 \in P \\ \Rightarrow \text{ and } q_0 \in Q^+ \text{ such that} \\ N(f-p_0/q_0) \leq \lambda. \end{array}$$

Existence Theorem. If  $N$  is a monotone norm of type (A) and if the triple  $(P, Q, N)$  has property (C), then each element  $f \in C[a,b]$  possesses a best approximation  $r^*$  in  $R$ ; that is, for any  $r \in R$ ,

$$N(f-r^*) \leq N(f-r).$$

Proof. Throughout the proof we use the notations  $\|f\|_S = \max_{x \in S} |f(x)|$  and  $\|f\|_\infty = \max_{a \leq x \leq b} |f(x)|$ . We begin by establishing the existence of two positive constants,  $\alpha$  and  $\beta$ , such that

(1)  $N_S(f) \leq \alpha \|f\|_S$  for all  $f \in C[a,b]$  and for all closed subsets  $S$  of  $[a,b]$ .

(2)  $N\left(\frac{p}{q}\right) \geq \beta \|p\|_\infty / \|q\|_\infty$  for all  $p/q \in R$ .

In order to prove (1), set  $\alpha = N(1)$ . From the monotonicity of  $N$ , we have  $\alpha \geq N_S(1)$ . Since  $|f(x)| \leq \|f\|_S$  for  $x \in S$ , it follows from the monotonicity of  $N_S$  that  $N_S(f) \leq N_S(\|f\|_S) = \|f\|_S N_S(1) \leq \alpha \|f\|_S$ . In order to prove (2), we make use of the fact that all norms on the finite dimensional subspace  $P$  are topologically equivalent.

Hence there exists a number  $\beta > 0$  such that  $N(p) \geq \beta \|p\|_\infty$  for all  $p \in P$ . Now for any  $p/q \in R$  we have  $\left| \frac{p(x)}{q(x)} \right| \geq \frac{|p(x)|}{\|q\|_\infty}$ . Hence by the monotonicity of  $N$ ,  $N\left(\frac{p}{q}\right) \geq N\left(\frac{p}{\|q\|_\infty}\right) = \frac{1}{\|q\|_\infty} N(p) \geq \beta \frac{1}{\|q\|_\infty} \|p\|_\infty$ .

We proceed now to the proof of the assertion in the theorem. Put  $\lambda = \inf_{r \in R} N(f-r)$ . Then there exists a sequence  $r_k \in R$  with the property  $N(f-r_k) \rightarrow \lambda$ . Put  $r_k = p_k/q_k$  with  $p_k \in P$  and  $q_k \in Q^+$ . There is no loss of generality in assuming that  $\|q_k\|_\infty = 1$ . Since the sequence  $N(f-r_k)$  is bounded the sequence  $N(r_k)$  is bounded. By inequality (2), the sequence  $\|p_k\|_\infty$  is bounded. Thus by passing to subsequences if necessary we may assume that the sequences  $p_k$  and  $q_k$  are uniformly convergent; say  $p_k \rightarrow p$  and  $q_k \rightarrow q$ . If  $q \in Q^+$  then the function  $r_0 = p/q$  has the property  $N(f-r_0) = \lambda$ . Indeed  $\|r_k - r_0\|_\infty \rightarrow 0$  whence, by inequality (1),  $N(r_k - r_0) \rightarrow 0$ . Thus from the inequality  $N(f-r_0) \leq N(f-r_k) + N(r_k - r_0)$  the desired conclusion follows. If  $q \notin Q^+$  then let  $S$  be any closed subset of  $[a,b]$  such that  $q(x) > 0$  on  $S$ . Clearly  $S$  is non-empty. Then  $r_k \rightarrow r = p/q$  uniformly on  $S$ . Consequently



$N_S(r_k - r) \rightarrow 0$  and  $N_S(f - r) \leq \lambda$  as before. By property (C), there exists  $p^* \in P$  and  $q^* \in Q^+$  such that  $N\left(f - \frac{p^*}{q^*}\right) \leq \lambda$ .

Example 1. Let  $N_S(f) = \max_{x \in S} |f(x)|$ . This is a monotone norm. Let  $P = [1, x, \dots, x^n]$  and  $Q = [1, x, \dots, x^m]$ , where the notation [...] denotes the linear span of the given set. For property (C), we note that if  $N_S\left(f - \frac{p}{q}\right) \leq \lambda$  whenever  $S$  is a closed set contained in the support of  $q$  then  $p/q$  can have no poles. That is,  $p/q$  can have at worst *removable* singularities. We may then write  $p/q = p^*/q^*$  with  $q^* > 0$  on  $[a, b]$ .

Example 2. Let  $N_S(f) = \max_{x \in S} |f(x)|$ ,  $P = [1, \cos x, \dots, \cos nx, \sin nx]$ , and  $Q = [1, \cos x, \sin x, \dots, \cos mx, \sin mx]$ . Property (C) is established by a Lemma of [1].

Example 3. Let  $N_S(f) = \left\{ \int_S |f(x)|^p w(x) dx \right\}^{1/p}$  with  $p \geq 1$ . This is clearly a monotone norm. Let  $P = [1, x, \dots, x^n]$  and  $Q = [1, x, \dots, x^m]$ . For property (C), suppose that  $N_S(f - p/q) \leq \lambda$  for every closed set  $S$  excluding the roots of  $q$ . Then  $p/q$  again may have only removable singularities, and thus can be replaced by an equivalent  $p^*/q^*$  with  $q^* \in Q^+$ .

Observe that for  $0 \leq p < 1$  this argument fails. For example,  $\int_{-1}^1 \left| \frac{1}{x} \right|^p dx = \frac{2}{1-p}$  when  $0 \leq p < 1$ , but the singularity cannot be removed.

It is easy to prove in any normed linear space that if  $p_0$  is a relative minimum point on a set  $M$  of the function  $\|f - p\|$ ,  $f$  being fixed, then  $p_0$  is a point of  $M$  closest to each point

$\theta = \lambda f + (1-\lambda)p_0$  for all sufficiently small positive  $\lambda$ . Specifically, suppose  $\|f - p_0\| \leq \|f - p\|$  for all  $p \in M$  such that  $\|p - p_0\| \leq K$ . If  $0 \leq \lambda \leq 1$  and  $\|\theta - p_0\| \leq K/2$ , then  $\|\theta - p_0\| \leq \|\theta - p\|$  for all  $p \in M$ .

Similarly, in an inner product space, if  $M$  is any subset, if  $f$  is a point outside  $M$ , and if  $p_0$  is a best approximation in  $M$  to  $f$ , then  $p_0$  is the *unique* best approximation in  $M$  to each point  $\lambda f + (1-\lambda)p_0$  for all  $\lambda \in [0,1]$ . We have no further information about the unicity question for best approximations from  $R(P,Q)$ .

We turn next to the "reverse problem" of approximation theory: if a set  $M$  and a point  $p \in M$  are prescribed, does there exist an  $f \neq p$  for which  $p$  is the best approximation in  $M$ ? In Hilbert space, this question is easily answered in the case that  $M^\perp \neq 0$ . Indeed if  $\langle v, x \rangle = 0$  for all  $x \in M$  then  $p$  is the best approximation in  $M$  to every point  $p + \lambda v$ . However, in the context of generalized rational approximation this argument will rarely succeed since generally  $R^\perp = 0$ , as we shall see presently.

It should also be noted that even in 2-space there exist smooth manifolds and points on them which are not closest points to exterior points. For example, the point  $(0,0)$  on the curve  $M = \{(x,y): y = x^{5/3}\}$  is not a closest point to an, outside point. The only candidate for such an outside point would be on the normal to the curve at the origin, and thus would be of the form  $(0,b)$ . But if  $b > 0$ , we select  $\lambda > 0$  so that  $\lambda + \lambda^5 < 2b$  and then find that the point  $(\lambda^3, \lambda^5)$  is chosen closer to

$(0,b)$  than is  $(0,0)$ .

Lemma. If  $\phi$  is continuous and strictly monotone on  $[a,b]$ , then the set  $\{1, \phi, \phi^2, \dots\}$  is fundamental in  $C[a,b]$ . Thus a nonzero  $f \in C[a,b]$  cannot have the property  $\int_a^b f \phi^n = 0$  for all  $n = 0, 1, 2, \dots$ .

Proof. Since  $\phi$  has an inverse, we can select (using the Weierstrass Theorem) a polynomial  $p$  to approximate  $f \cdot \phi^{-1}$ . Thus  $|f(x) - p(\phi(x))| < \epsilon$  for all  $x$  in  $[a,b]$ . If  $\int_a^b f \phi^n = 0$  for all  $n$  then

$$\int_a^b f^2 = \int_a^b f(f - p \cdot \phi) \leq \epsilon (b-a) \|f\|_{\infty}.$$

Hence  $\int_a^b f^2 = 0$  and  $f = 0$ .

Theorem. If  $Q^+$  contains two elements whose ratio is strictly monotone on  $[a,b]$  and if  $P$  contains an element whose support is dense in  $[a,b]$  then the orthogonal complement of  $R(P,Q)$  in  $C[a,b]$  is 0.

Proof. By making a linear change of variable we may take the interval  $[a,b]$  to be  $[-1,1]$ . Select  $q_0$  and  $q_1$  in  $Q^+$  in such a way that the function  $\phi = q_1/q_0$  is strictly monotone. There is no loss of generality in supposing that  $\|\phi\|_{\infty} < 1$ . Select an element  $p_0 \in P$  with dense support. For each  $\lambda \in [-1,0]$ , we have  $q_0 - \lambda q_1 \in Q^+$ , and the series  $\frac{1}{1 - \lambda \phi(x)} = \sum_{n=0}^{\infty} [\lambda \phi(x)]^n$  is uniformly convergent for  $-1 \leq x \leq 1$ . Consequently, if  $f$  is an element of  $C[a,b]$  orthogonal to  $R$ , then

$$\begin{aligned} 0 &= \int_{-1}^1 f \frac{p_0}{q_0 - \lambda q_1} w = \int_{-1}^1 \frac{fp_0 w}{q_0} \frac{1}{1 - \lambda \phi} = \int_{-1}^1 \frac{fp_0 w}{q_0} \sum_{n=0}^{\infty} (\lambda \phi)^n \\ &= \sum_{n=0}^{\infty} \lambda^n \int_{-1}^1 \frac{fp_0 w}{q_0} \phi^n. \end{aligned}$$

Since this power series in  $\lambda$  vanishes for  $\lambda \in [-1, 0]$ , its coefficients must vanish:

$$\int_{-1}^1 \frac{fp_0 w}{q_0} \phi^n = 0 \quad n = 0, 1, 2, \dots$$

By the preceding lemma,  $fp_0 w/q_0 = 0$ , and by the hypotheses concerning  $p_0$ ,  $q_0$ , and  $w$ , it follows that  $f = 0$ .

Corollary. The orthogonal complement of  $R_m^n[a, b]$  in  $C[a, b]$  is 0 whenever  $n \geq 0$  and  $m \geq 1$ .

The hypothesis on an element  $r = p/q \in R$  that

$$\dim(pQ + qP) = \dim P + \dim Q - 1$$

arises frequently in this subject. Such elements of  $R$  were termed *normal* elements in [1]. For  $r = p/q \in R_m^n[a, b]$ , with  $p$  and  $q$  relatively prime, normality is equivalent to

$$\min\{n - \partial p, m - \partial q\} = 0$$

where  $\partial$  means degree of, and  $P$  is of dimension  $n+1$  and  $Q$  is of dimension  $m+1$ . This was proved in [1]. See below, where this assertion occurs as a corollary of another result. It is also known that the normal elements of  $R$  are precisely the elements  $p/q$  where the mapping

$(p,q) \rightarrow p/q$  is topological from  $P \oplus Q^{(1)}$  into  $C[a,b]$ . Here  $Q^{(1)} = \{q \in Q^+ : \|q\| = 1\}$ . See [7]. The condition of normality also implies that  $\dim(pQ \cap qP) \leq 1$ . See [7].

The following is familiar from calculus:

Lemma. Let  $\phi$  be a real-valued function defined on an open set  $D$  in  $n$ -space. Assume that the  $2^{\text{nd}}$  partial derivatives of  $\phi$  are continuous in a neighborhood of a point  $x_0 \in D$ . Let  $S$  be any subset of  $n$ -space. Then the condition

$$\left. \begin{array}{l} 0 \neq h \in S \\ x_0 + h \in D \end{array} \right\} \Rightarrow \left. \begin{array}{l} \frac{d}{d\lambda} \phi(x_0 + \lambda h) \Big|_{\lambda=0} = 0 \\ \frac{d^2}{d\lambda^2} \phi(x_0 + \lambda h) \Big|_{\lambda=0} > 0 \end{array} \right\}$$

implies the existence of an  $\epsilon > 0$  with the property

$$\left. \begin{array}{l} 0 < |h| < \epsilon \\ h \in S \\ x_0 + h \in D \end{array} \right\} \Rightarrow \phi(x_0) < \phi(x_0 + h).$$

Theorem. In the generalized rational approximation problem, let  $r_0 \equiv p_0/q_0$  be a normal element of  $R$ . Then  $r_0$  is a best  $L^2$ -approximation to some  $f \in C[a,b] \sim R$ .

Proof. In the linear space  $P \oplus Q$  we define the linear subspace  $T = \{(p,q) : pq_0 = p_0q\}$ . Let  $S$  be the orthogonal complement of  $T$ ; thus  $P \oplus Q = T \oplus S$ . We observe that  $\dim T = 1^*$ . Let  $D$  be the open set  $\{(p,q) : q > 0\}$ . On  $D$  define  $\phi(p,q) = \int \left(\frac{p}{q} - f\right)^2 w$ , in which  $f$  is to be selected in such a way that the hypotheses of the preceding

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\* A proof of this may be found in [7], Theorem 1.

lemma are satisfied by  $D$ ,  $S$ , and  $\phi$  as just defined. Namely, we wish to secure the conditions

$$\left. \begin{array}{l} 0 \neq (p,q) \in S \\ (p_0, q_0) + (p,q) \in D \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \frac{d}{d\lambda} \phi(p_0 + \lambda p, q_0 + \lambda q) \Big|_{\lambda=0} = 0 \\ \frac{d^2}{d\lambda^2} \phi(p_0 + \lambda p, q_0 + \lambda q) \Big|_{\lambda=0} > 0. \end{array} \right.$$

After effecting the differentiations and setting  $\lambda = 0$  we arrive at the conditions

$$\left. \begin{array}{l} 0 \neq (p,q) \in S \\ q_0 + q > 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \int \frac{r_0 - f}{q_0^2} (q_0 p - p_0 q) w = 0 \\ \int \frac{(q_0 p - p_0 q)^2}{q_0^4} w - \frac{2(r_0 - f)q(q_0 p - p_0 q)}{q_0^3} w > 0. \end{array} \right.$$

In order to satisfy the first condition, we select  $0 \neq h \perp (q_0 p + p_0 q)w$  and set  $f = r_0 + \frac{1}{2}q_0^2 h$ . The next step in the proof is to show that if  $||h||$  is sufficiently small, the second condition will be met also.

Consider the linear operator  $L: L(p,q) \rightarrow q_0 p - p_0 q$ . Its null space is  $T$ . Hence, by a familiar result in linear algebra, for an arbitrary norm,  $||\cdot||$ , there exists a constant  $\alpha > 0$ , such that  $||L(p,q)|| \geq \sqrt{\alpha} \text{ dist}[(p,q), T]$ . Hence, for  $(p,q) \in S$  we have  $||L(p,q)||^2 \geq \alpha(||p||^2 + ||q||^2)$ , so that  $\int \left[ \frac{(q_0 p - p_0 q)^2}{q_0^4} + \frac{qh(q_0 p - p_0 q)}{q_0} \right] w \geq \beta(||p||^2 + ||q||^2) - \gamma||q|| ||h|| \sqrt{||p||^2 + ||q||^2} \geq \sqrt{||p||^2 + ||q||^2} \left[ \beta \sqrt{||p||^2 + ||q||^2} - \gamma||q|| ||h|| \right]$ ; here norms with weights  $\frac{w}{q_0}$  and  $\frac{w}{q_0}$  respectively, have been employed to yield the constants  $\beta$  and  $\gamma$ . Now if  $||h|| < \frac{\beta}{\gamma}$  then this last quantity is positive.

By the lemma,  $\phi(p_0, q_0) < \phi(p_0+p, q_0+q)$  whenever the pair  $(p, q)$  satisfies  $0 < ||p||^2 + ||q||^2 < \epsilon$ ,  $q_0 + q > 0$ , and  $(p, q) \in S$ . We now show that  $(p_0, q_0)$  is a local minimum point for  $\phi$  in  $D$ . If necessary, we decrease  $\epsilon$  so that  $(p_0+p, q_0+q) \in D$  whenever  $||p||^2 + ||q||^2 < \epsilon$ . Let  $(p, q)$  be an arbitrary pair satisfying  $||p||^2 + ||q||^2 < \epsilon$ . Select  $\lambda$  so that the projection of  $\lambda(p_0+p, q_0+q)$  onto  $T$  is  $(p_0, q_0)$ . Thus  $\lambda(p_0+p, q_0+q) = (p_0, q_0) + (p_1, q_1)$  with  $(p_1, q_1) \in S$ . It is readily verified that  $||p_1||^2 + ||q_1||^2 < \epsilon$ . Thus  $\phi(p_0+p, q_0+q) = \phi(\lambda p_0 + \lambda p, \lambda q_0 + \lambda q_1) = \phi(p_0+p_1, q_0+q_1) > \phi(p_0, q_0)$  by the lemma. Finally, by an earlier remark,  $p_0/q_0$  is a closest point to some functions of the form  $\lambda f + (1-\lambda)p_0/q_0$ , provided that  $\lambda$  is sufficiently small and positive.

Theorem. A non-normal element of  $R_m^n[a, b]$  cannot be a best  $L^2$ -approximation to an  $f \in C[a, b] \sim R_m^n[a, b]$ .

Proof. Suppose that  $r = p/q$  is non-normal in  $R_m^n[a, b]$ . Then  $\delta p < n$  and  $\delta q < m$ . Suppose that  $r$  is a best  $L^2$ -approximation to some  $f \in C[a, b]$ . We are going to prove that  $f = r$ . Let  $s$  denote any function  $s(x) = x - \alpha$  where  $\alpha \notin [a, b]$ . Then for all real  $\lambda$  the function  $r_\lambda = r + \frac{\lambda}{qs}$  belongs to  $R_m^n[a, b]$ . If we form the function  $\phi(\lambda) = \int (r_\lambda - f)^2 w$  then  $\phi'(0) = 0$  because  $r$  is a best approximation to  $f$ . But  $\phi'(0) = 2 \int (r-f) \frac{w}{qs}$ . Since this vanishes for all  $s$ , we see that  $\frac{(r-f)w}{q}$  lies in the orthogonal complement of  $R_1^0$  and must vanish, in accordance with a theorem proved above.

Because there are elements arbitrarily close to  $f$  with unique

best approximation out of  $R_m^n[a,b]$  we have the

Corollary. Given  $f \in C[a,b]$  define  $\xi_m^n(f) = \inf_{r \in R_m^n[a,b]} \|f - r\|_2$ .

If  $f$  is not a rational function, then  $\xi_m^n(f) > \xi_{m+1}^{n+1}(f)$  for all  $n$  and  $m$ .

Theorem. Let  $r_0 = p_0/q_0$  be a local minimum point of  $\|r - f\|_2$  then

$$(1) \quad \int (f - r_0)(p_0 q + q_0 p) \frac{w}{q_0^2} = 0 \quad \text{for all } (p, q)$$

$$(2) \quad \int [p^2 + 2(f - 2r_0)qp + (3r_0^2 - 2fr_0)q^2] \frac{w}{q_0^2} \geq 0 \quad \text{for all } (p, q)$$

Proof. For any pair  $(p, q)$  we define  $\phi(\lambda) = \int \frac{p_0 + \lambda p}{q_0 + \lambda q} - f^2 w$ .

In order for  $r_0$  to be a local minimum point of  $\|r - f\|_w^2$  it is necessary that 0 be a local minimum point of  $\phi$ . Hence we must have  $\phi'(0) = 0$  and  $\phi''(0) \geq 0$ , which imply (1) and (2).

For any  $n = 0, 1, 2, \dots$  let  $P_n$  denote the linear space of all real polynomials having degree  $\leq n$ . The degree of a polynomial  $p$  is denoted by  $\delta p$ .

Lemma. If  $p \in P_n$ ,  $q \in P_m$ , and if the pair  $(p, q)$  is relatively prime, then

$$pP_m + qP_n = P_k$$

with  $k = \max\{n + \delta q, m + \delta p\}$ .

Proof. In order to prove that  $pP_m + qP_n \subset P_k$  it is enough to observe that if  $f \in pP_m \oplus qP_n$  then  $f$  is a polynomial whose degree



does not exceed  $\max\{p + m, q + n\}$ , and hence  $f \in P_k$ .

For the inclusion  $P_k \subset pP_m \oplus qP_n$  we distinguish two cases according to whether  $k = n + \partial q$  or  $k = m + \partial p$ . By considerations of symmetry it suffices to discuss the case  $k = n + \partial q$ . By the fact that the pair  $(p, q)$  is relatively prime, there exist polynomials  $s$  and  $t$  such that  $1 = sp + tq$ . If  $f$  is an arbitrary element of  $P_k$ , we have  $f = fsp + ftq$ . By the division algorithm we write  $fs = uq + r$  where  $\partial r < \partial q$ . Then  $f = (uq + r)p + ftq = rp + (up + ft)q$ . Since  $\partial f \leq k$  and  $\partial(rp) = \partial r + \partial p < \partial q + m = k$ , it follows that  $\partial[(up + ft)q] \leq k = n + \partial q$ . Hence  $\partial(up + ft) \leq n$ . We have therefore shown that  $f \in pP_{m-1} \oplus qP_n \subset pP_m \oplus qP_n$ .

Corollary. An element  $r \in R_m^n$  is normal if and only if its irreducible representation  $p/q$  has the property  $\min\{n - \partial p, m - \partial q\} = 0$ .

An  $n$ -dimensional subspace  $M$  in  $C[a, b]$  is called a *Haar* subspace if  $0$  is the only member of  $M$  which has more than  $n-1$  roots. It is known (see [8]) that if  $M$  is an  $n$ -dimensional Haar subspace and if  $0 \neq f \in C[a, b] \cap M^\perp$  then  $f$  changes sign at least  $n$  times on  $(a, b)$ . From this we obtain the following result.

Theorem. Let  $r_0 \equiv p_0/q_0$  be a best  $L^2$ -approximation in  $R$  to a function  $f \neq r_0$ . If  $p_0Q + q_0P$  is a Haar subspace of dimension  $k$ , then  $r$  interpolates to  $f$  in at least  $k$  points.

Proof. One of the necessary properties of a best approximation was established earlier as

$$\int_a^b (f - r_0)(p_0q + q_0p) \frac{w}{q_0^2} = 0 \quad (p \in P, q \in Q).$$

Hence  $f - r_0$  changes sign at least  $k$  times by the preceding remark.

Corollary. Let  $r_0$  be a best  $L^2$ -approximation in  $R_m^n[a,b]$  to  $f$ . Then  $r_0$  interpolates to  $f$  in at least  $n + m + 1$  points.

Proof. If  $f = r_0$  there is nothing to prove. Hence assume that  $f \neq r_0$ . We have seen earlier that  $r_0$  must be normal. Hence  $p_0Q + q_0P$  is the space of all polynomials of degree  $\leq n+m$ . By the preceding theorem,  $f - r_0$  changes sign in at least  $n + m + 1$  points.

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